DYNAMICS AND THRESHOLDS OF A SIMPLE EPIDEMIOLOGICAL MODEL: EXAMPLE OF HIV/AIDS IN MALI

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ABSTRACT. The dynamic of many epidemiological models for infectious diseases that spread in the sexually active population present a crucial period: the period of the influx or recruitment of susceptible. In this paper we assume that the recruitment of susceptible is done among the juvenile group. We propose a dynamical system to modelize the disease spread and we study the dynamical behavior of this system. Then, the controllability of the system is studied. We prove that the survival rate allows to control the dynamic of the system. Numerical simulations are given to illustrate the results.

1. INTRODUCTION

In recent years several authors have described interesting dynamical behavior of *SIR* epidemiological models in which the population can be portioned into two age structured classes: immature individuals and mature ones (see for example [3]). The HIV disease belongs to the class of diseases which spread essentially among sexually active individuals. Thus, it is meaningful to consider stage structure in epidemiological models. The population is initially divided into two compartments: those, who are mature individuals or adults and those who are in youthful age or immature individuals. All population groups are subject to the risk of dying from AIDS.
We denote by:

- $J(t)$ the density of the immature individuals;
- $M(t)$ the density of the mature individuals;
- r_1 the survival rate of the immature individuals;
- r_2 the survival rate of the mature individuals;
- $B(t)$ the birth density in the population;
- m the rate of immature individuals becoming mature individuals.

Then the discrete single population model with stage structure reads:

$$\left\{\begin{array}{rcl} J(t+1) & = & B(t)+r_1J(t)-r_1mJ(t), \\ M(t+1) & = & r_1mJ(t)+r_2M(t) \\ N(t) & = & J(t)+M(t). \end{array}\right. \tag{1.1}$$

For describing the disease transmission, a traditional *SIR* model is introduced. Each member of the population is considered to belong to one of the three classes: Susceptible individuals (denoted by S), Infected individuals (denoted by I) and Removed individuals (denoted by R). Each individual begins in the class S, only to move to the class I after coming into contact with an infected person. Infected individuals eventually recover from the disease due to a medical treatment and then move to the class R and are unable to be infected one

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again. The disease is fueled by supply of new susceptibles issued from the compartment J. The size of the population is denoted by $N(t)$ and can be expressed as the following sum

$$N(t) = S(t) + I(t) + R(t) + J(t). \tag{1.2}$$

The *SIR* model reads:

$$\begin{cases} \dfrac{dS}{dt} &= r_1 m(1-\tau)J(t) - F_i(I,t)S + r_2 S \\ \dfrac{dI}{dt} &= F_i(I,t)S - r_3(\sigma+\alpha)I \\ \dfrac{dR}{dt} &= r_3\sigma I - \mu R \end{cases} \tag{1.3}$$

where:

- $F_i(I,t)$ is the incidence function which may vary periodically. It is usual to take $F_i(I,t) = \Omega(t)I$ in which $\Omega(t)$ is the transmission rate; it is either constant, or a periodic modulation about a constant value, for example $\Omega(t) = \Omega_0(1+\Omega_1 sin(\omega t))$;
- r_3 is the survival rate of the infected mature individuals and recovered with the probability σ;
- α is the rate of death due to the disease;
- τ is the rate of transmission from mother to child;
- μ is the rate of death due to other causes.

The aim of this work is to provide simple conditions for the parameters of the SIR model (1.3) that makes possible to control the infected individuals. By using the notion of the exterior contingent cone to a convex subset C of $\mathbb{R}^2$, we prove that the system (1.3) is controllable with three of its parameters. Whatever the initial conditions are, the system (1.3) reaches the subset C and remains in C. The paper is organized as follows: the introduction ends with an existence and uniqueness result. In sections 2 the controllability of the system (1.3) is studied and several numerical results are presented in connection with available data concerning Mali.

The dynamic behavior of (1.3) is determined by the variation of I and R . According to (1.2) the suceptibles compartment is expressed as $S(t) = N(t) - I(t) - R(t) - J(t)$, thus (1.3) is reduced to:

$$\begin{cases} \dfrac{dI}{dt} &= F_i(I,t)[N(t) - J(t) - R(t) - I(t)] - r_3(\sigma+\alpha)I \\ \dfrac{dR}{dt} &= r_3\sigma I - \mu R. \end{cases} \tag{1.4}$$

Since $\mu > 0$ a new timescale $t^{'} = \mu t$ is introduced. System (1.4) becomes:

$$\begin{cases} \dfrac{dI}{dt'} &= \Omega(t^{'})I(N-J-R-I) - r_3^{'}(\sigma+\alpha)I \\ \dfrac{dR}{dt'} &= r_3^{'}\sigma I - R. \end{cases} \tag{1.5}$$

We assume that $\Delta = N(t) - J(t) - R(t)$ is constant. Defining $\gamma = r_3'\sigma$, and omitting the prime notations, the system (1.5) becomes:

$$\left\{ \begin{array}{rcl} \dfrac{dI}{dt} & = & \Omega(t)I(\Delta - I) - \gamma I - \dfrac{\gamma\alpha}{\sigma}I \\ \dfrac{dR}{dt} & = & \gamma I - R. \end{array} \right. \tag{1.6}$$

Theorem 1.1. *Assume Ω to be $C^1(\mathbb{R}_+;\mathbb{R})$ function a primitive of which is bounded. For every initial condition $(I^*, R^*) \in \mathbb{R}_+^2$, the solution $(I(\cdot), R(\cdot)) : \mathbb{R}_+ \to \mathbb{R}_+^2$ to (1.6) belongs to K a compact subset of $\mathbb{R}_+^2$.*

Proof. Set $\Theta = \gamma(1 + \dfrac{\alpha}{\sigma})$, by integrating the first equation of (1.6) we have

$$I(t) = \frac{I_0 e^{\int_0^t (\Omega(\tau)\Delta - \Theta)\,d\tau}}{1 + \displaystyle\int_0^t I_0 e^{\int_0^t (\Omega(\tau)\Delta - \Theta)\,d\tau} ds}.$$

Let M be a bound from below of a primitive of Ω, we have:

$$0 \le I(t) < \frac{I_0 e^{M\Delta t}}{e^{\Theta t}(1 + \displaystyle\int_0^t I_0 e^{\int_0^t (\Omega(\tau)\Delta - \Theta)\,d\tau} ds)} = \bar{I}. \tag{1.7}$$

From $\dfrac{dR}{dt} = \gamma I - R$ we deduce:

$$\begin{aligned} R(t) &= R_0 e^{-t} + \gamma e^{-t} \int_0^t e^s I(s) ds \\ &\le R_0 e^{-t} + \gamma e^{-t}(e^t - 1)\bar{I} \\ &< \quad R_0 e^{-t} \gamma \bar{I} = \bar{R}. \end{aligned}$$

So the Poincaré-Bendixson's theorem [6] claims : either the solution (I, R) to the system (1.6) tends to or is a critical point when time goes to infinity, or it tends to or it is a periodic solution.

A complete bifurcation analysis is beyond the scoop of this paper. For a precise study of the orbits the reader is refereed to [7] or [13] for example.

2. Controlability of the model with its coefficients

The question we address in this section reads: do exist parameters which allow the system (1.6) to evolve towards a fixed region C of the plan (I, R) whatever the initial conditions are. For $0 < \underline{x}_1$ fixed, we define the convex domain C of the plan and its associated truncated cylinder C_T by:

$$\begin{array}{c} C = \{(x_1, x_2) \in \mathbb{R}_+^2; \quad x_1 \le \underline{x}_1; \text{ and } \frac{3}{4}x_1 \le x_2\}; \\ C_T = \{(t, x_1, x_2) \in \mathbb{R}_+^3; \quad 0 \le t \le T; \quad x_1 \le \underline{x}_1; \text{ and } \frac{3}{4}x_1 \le x_2\} \end{array} \tag{2.1}$$

Definition 2.1. (contingent and exterior contingent cone). The contingent cone to C_T at x $T_{C_T}(x)$ is constituted by vectors $v \in \mathbb{R}^3$ verifying:

$$\lim_{h\to 0^+} \inf \frac{d_{C_T}(x+hv, C_T)}{h} = 0,$$

where d_{C_T} denotes the distance to the subset C_T. The exterior contingent cone $T_{C_T}(x)$ is constituted by vectors $v \in \mathbb{R}^3$ verifying:

$$\lim_{h\to 0^+} \inf \frac{d_{C_T}(x+hv, C_T) - d_{C_T}(x)}{h} \leq 0,$$

When a point x belongs to the boundary of C_T the definition of exterior contingent cone is equivalent to the definition of the contingent cone. We have the following result [12] (Theorem 3.4.1 p. 102).

Lemma 2.2. *The exterior contingent cone to C_T at point x is constituted by vectors $v \in \mathbb{R}^3$ verifying:*

$$(x - P_{C_T}x, v) \leq 0;$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product, and P_{C_T} stands for the orthogonal projection on C_T.

Before stating the result of controllability, we give some technicalities. Set

$$F(t,x_1,x_2) = \begin{pmatrix} 1 \\ \Omega(t)x_1(\Delta - x_1) - x_1\gamma(1+\dfrac{\alpha}{\sigma}) \\ \gamma x_1 - x_2 \end{pmatrix}, \text{ we have:} \tag{2.2}$$

Lemma 2.3. *Let $X \in \{(t,x_1,x_2), 0 < t < T; 0 < x_1; 0 < x_2\} \cap C_T^c$ be fixed. Then $X - P_{C_T}X$ is the outward normal to C_T whenever it exists, and for $0 \leq s \leq 1$ is given by :*

$$X - P_{C_T}X = \begin{pmatrix} 0 \\ 1 \\ -\frac{4}{3}s \end{pmatrix}.$$

Furthermore, a sufficient condition for the vector $F(X)$ to belong to the exterior contingent cone T_{C_T} reads:

$$x_1\Big[\Omega(t)(\Delta - x_1) - \gamma(1+\frac{\alpha}{\sigma}) + 1)\Big] \leq 0. \tag{2.3}$$

Proof. From the definition of the exterior contingent cone T_{C_T} we have:

$$\forall s \in [0,1], -\Omega(t)x_1^2 + x_1\big(\Omega(t)\Delta - \gamma(1+\frac{\alpha}{\sigma}) - \frac{4s}{3}\big) \leq \frac{4s}{3}x_2. \tag{2.4}$$

A sufficient condition independent of s for condition (2.4) to be satisfied is obtained when $x_2 \leq \dfrac{3}{4}x_1$ with $0 \leq x_1$ and reads:

$$-\Omega(t)x_1^2 + x_1\Big(\Omega(t)\Delta - \gamma(1+\frac{\alpha}{\sigma})\Big) \leq -x_1. \tag{2.5}$$

□

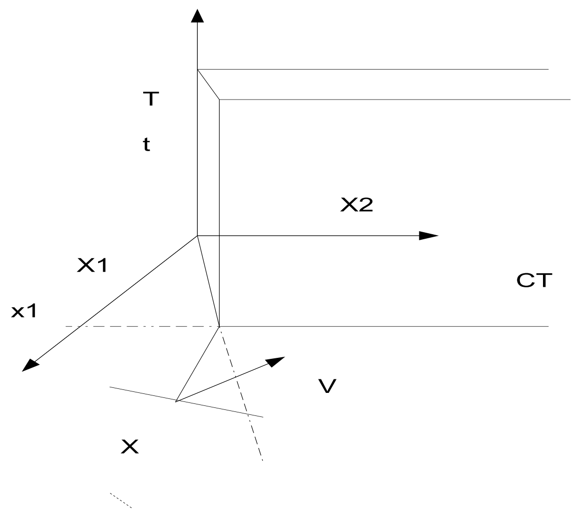

FIGURE 1. Exterior contingent cone

Theorem 2.4. *Let be* $0 \leq \max_{0 \leq t} \Omega(t) = \overline{\Omega}$*, and let parameters* $\alpha, 0 < \underline{x}_1 < \Delta, \sigma$ *be fixed. Whatever* $(I_0, R_0) \in \mathbb{R}^2$ *are, choose* r_3' *in such a way that* $\gamma = r_3'\sigma$ *verifies:*

$$0 < (\gamma - \frac{3}{4}); \quad \overline{\Omega}\big(\Delta - \underline{x}_1\big) - \gamma\big(1 + \frac{\alpha}{\sigma}\big) + 1 \leq 0. \tag{2.6}$$

Then, there exists $0 \leq T_r$ *such that for all time* t *greater or equal than* T_r*, the solution* $(I(t), R(t))$ *to problem (1.6) belongs to the subset* C*.*

Proof. Set $Y = (t, I(t), R(t))$, then Problem (1.6) is expressed as the following autonomous system:

$$\begin{cases} \dfrac{dY(t)}{dt} = F(Y(t)); \quad 0 < t \\ Y(0) = (0, I_0, R_0). \end{cases} \tag{2.7}$$

Define the function $G(t, I)$ by:

$$G(t, I) = \Omega(t)(\Delta - I) - \gamma(1 + \frac{\alpha}{\sigma}) + 1.$$

Function G is a decreasing function with respect to I for all time. Thus if $G(t, x_1) \leq 0$, it will be negative for all $I > x_1$. Condition (2.6) implies that $F_2(t, I, R)$ is negative and $F_3(t, I, R)$ is positive for all $0 < t$; $x_1 < I$; $0 \leq R$. Theorem 1.1 asserts the existence of $(I(t), R(t))$ for all time t. A simple continuity argument implies that the subset C defined in (2.1) is reached for a time T_r by the trajectory starting at the point (I_0, R_0). Fix $T > T_r$, Lemma 2.2 claims that condition (2.6) is a sufficient condition for $F(Y) \in T_{C_T}(Y)$ when Y belongs to the boundary of C_T. Nagumo's theorem applies for Equation (2.7) with initial conditions $(T_r, I(t_r), R(T_r))$, and we get that $(I(t), R(t)) \in C$ for $T_r \leq t$ ([1] Theorem 1 P. 27). $\square$

As consequence of Theorem 2.4 the SIR models allow to evaluate medical policies with precise aims. The sufficient condition (2.6) characterizes the treatment effort through the survival rate r_3 of the infected mature individuals recovered with the probability σ.
Let us end this section with numerical examples. By using available data from Mali, we

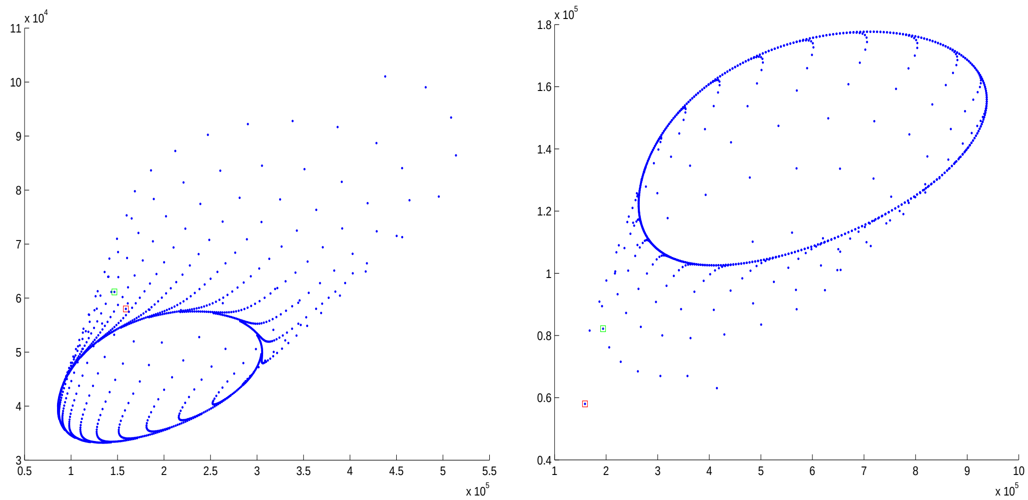

FIGURE 2

(a) $\overline{\Omega}(\Delta - \underline{x}_1) - \gamma(1 + \frac{\alpha}{\sigma}) \leq -1$ **(b)** $\overline{\Omega}(\Delta - \underline{x}_1) - \gamma(1 + \frac{\alpha}{\sigma}) + 1 > 0$

have: $\Delta = 6066573$; and we fix $\gamma = 0.25$ and $\alpha = 0.01$. $\underline{x}_1 = 1.5\,10^5$. The initial conditions are $(I_0, R_0) = (1.59\,10^5, 5.8\,10^4)$. In FIGURE 2 we have considered the case (a) with $\Omega = 4.3\,10^{-8}$; $\sigma = 0.5$. The sufficient condition (2.6) is not satisfied, nevertheless, it can be checked that after a long time the computed solution $(I(t), R(t))$ represented with aa green point has a first component less or equal than $\overline{x}_1$. In case (b), we have $\Omega = 4.6\,10^{-8}$; $\sigma = 0.8$; The sufficient condition (2.6) is not satisfied. Here all the trajectories are outside of the cone C_T.

The cone C_T, roughly speaking, characterizes the treatment effort. The sufficient condition (2.6) is basically governed by tow parameters: the transmission rate, and the survival rate r_3 of the infected mature individuals recovered with the probability σ. In the following examples, keeping the same values for parameters as in Case (a) except for r_3. For $r_3 = 0.85$, The sufficient condition (2.6) is not satisfied, and we have in Figure 3 left all the trajectories outside of C_T. For $r_3 = 2$, The sufficient condition (2.6) is satisfied, and the trajectories are concentrated in a neighborhood of the disease-free equilibrium $(0,0)$ see Figure 3 right.

In this paper, it is shown that by using the exterior contingent cone and a viability theorem, simple convex subsets are reachable with a SIR model by adjusting some coefficients. Thus, it will be possible to predict with a certain accuracy the evolution level of the disease by changing one or another of parameters. In our example, it is attractive to see that, if the survival rate r_3 attains 2%, the disease almost goes back at a level disease-free equilibrium.

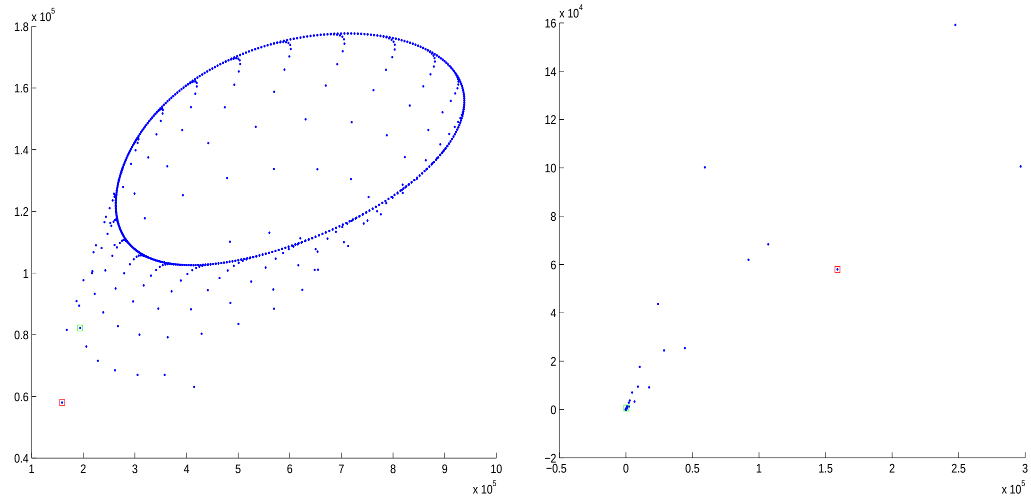

FIGURE 3

(a) $\frac{3}{4}x_1 \leq x_2$ is not satisfied with $r_3 = 0.85$; **(b)** $\frac{3}{4}x_1 \leq x_2$ is satisfied with $r_3 = 2$.

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